JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 4, November 2014
http://dx.doi.org/10.14403/jcms.2014.27.4.675

PRECONDITIONED GL-CGLS METHOD USING REGULARIZATION PARAMETERS CHOSEN FROM THE GLOBAL GENERALIZED CROSS VALIDATION

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Abstract. In this paper, we present an efficient way to determine a suitable value of the regularization parameter using the global generalized cross validation and analyze the experimental results from preconditioned global conjugate gradient linear least squares(Gl-CGLS) method in solving image deblurring problems. Preconditioned Gl-CGLS solves general linear systems with multiple right-hand sides. It has been shown in [10] that this method can be effectively applied to image deblurring problems. The regularization parameter, chosen from the global generalized cross validation, with preconditioned Gl-CGLS method can give better reconstructions of the true image than other parameters considered in this study.

1. Introduction

Regularized deblurring problems, illustrating general space-invariant imaging system, are often modeled as a linear least squares problem:

$$\min_x(\|Hx-b\|_2^2+\lambda^2\,\|x\|_2^2), \tag{1.1}$$

where $H_{N\times N}$ is a blurring ill-conditioned matrix with some block structures, b and x represent the observed and the original image respectively. The positive regularization parameter λ specifies the amount of regularization and, in general, an appropriate value of this parameter is not known a priori.

Received September 17, 2014; Accepted October 06, 2014.
2010 Mathematics Subject Classification: Primary 65F22, 65K10.
Key words and phrases: Tikhonov regularization, SVD, GCV, preconditioned Gl-CGLS, image restoration.
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This work was supported by research fund of 2014 Chungnam National University.

The regularization parameter λ controls the weight given to minimization of the regularization term, relative to the minimization of the residual norm. The regularization parameter plays a crucial role in the quality of the solution and thus an appropriate choice of the regularization parameter is important to the regularization method. There are various techniques to choose the approximate regularization parameters such as Morozov's discrepancy principle, L-curve criterion, and generalized cross valication(GCV)[3, 4, 5]. Each of these approaches has its individual advantages and disadvantages. Especially the GCV method is prominent for the selection of the crucial regularization parameters since GCV has good asymptotic properties for large number of noisy data. In iterative Lagrange method, the Lagrange multiplier also acts as a regularization parameter in the Fast Lagrange method for image restoration problems with reflective boundary conditions which give better reconstructions of the true image([9]).

Since the implementations of image restoration problems typically require the need of formidable data, we generalized the problem (1.1) to the following minimization problem with respect to the Frobenius norm

$$\min_{X}(\|HX - B\|_F^2 + \lambda^2 \|X\|_F^2), \tag{1.2}$$

where $B_{N\times s}(N \gg s)$ is a collection of the column stacking of each small blocks obtained from partitioning the blurred and noisy image ([1, 8, 10]).

In [10] we applied preconditioned global conjugate gradient linear least squares(Gl-CGLS) method to the image deblurring problems (1.2) with reflective boundary conditions and obtained that the Gl-CGLS approach can significantly improve execution time.

Since the appropriate choice of the regularization parameter in Gl-CGLS method improves the quality of solution to the regularization problem, we adapt and extend the prominent GCV technique to the problem (1.2) in this paper. GCV, whose basic idea is that a good choice of λ should predict missing values of the data, is a predictive statistics-based method that does not require a priori estimates of the error norm.

This paper is discussed in the following dimensions: The brief description of the generalized cross validation method for the problem (1.1) is summarized in Section 2. In Section 3, we present an appropriate global generalized cross validation for the problem (1.2). Section 4 illustrates an analysis of the extended global generalized cross validation implemented for image deblurring problems with multiple right hand sides. Numerical experiments and final remarks are described in Section 5.

2. Review of GCV

In this section, we recall generalized cross validation(GCV) as a chosen method for the regularization parameter in (1.1) (see [3, 4, 5, 6]). The GCV, as an noise-free method, is a predictive method to minimize the predictive mean-square error $\|Hx_\lambda - b_{exact}\|_2$. Since b_{exact} is unknown, we want to estimate the regularization parameter λ so that Hx_λ predicts b_{exact} as accurate as possible by using the cross validation process.

The cross validation separates the given data into two sets and uses one of the sets to compute an approximate solution to a reduced problem. The approximate solution is then used to predict the elements in the other set. Eliminating b_i, the ith element of b, we compute the Tikhonov solution $x_\lambda^{(i)}$ of the reduced problem:

$$x_\lambda^{(i)} = \left((H^{(i)})^T H^{(i)} + \lambda^2 I\right)^{-1} (H^{(i)})^T b^{(i)},$$

where $H^{(i)}$ and $b^{(i)}$ are the shortened versions of H and b with the ith row and element eliminated respectively. Then we can use $x_\lambda^{(i)}$ to predict the element b_i eliminated corresponding to the missing row of H by $H(i,:)x_\lambda^{(i)}$. The goal is then to determine the regularization parameter λ that minimizes the prediction errors for all data elements:

$$\lambda = \arg\min \frac{1}{N} \sum_{i=1}^{N} (H(i,:)x_\lambda^{(i)} - b_i)^2.$$

This is a formidable computation task since N number of different Tikhonov problems are involved. However, using some technical arguments to eliminate the element of b, one can show that the above minimization problem can be replaced by

$$\lambda = \arg\min \frac{1}{N} \sum_{i=1}^{N} \left(\frac{H(i,:)x_\lambda - b_i}{1 - h_{ii}}\right)^2,$$

where x_λ is the Tikhonov solution, and h_{ii} is the ith diagonal element of the matrix $H(H^T H + \lambda^2 I)^{-1} H^T$. Since the value of h_{ii} depends on the permutation of the rows of H arising from the ordering of the data, we replace each diagonal element h_{ii} with the average of the diagonal elements which leads to the generalized cross validation(GCV) as in the definition 2.1. The regularization parameter λ can be chosen from the following minimization problem

$$\lambda = \arg\min \frac{1}{N}\sum_{i=1}^{N}\left(\frac{H(i,:)x_\lambda - b_i}{1-\mathrm{trace}(H(H^TH+\lambda^2 I)^{-1}H^T)/N}\right)^2, \tag{2.1}$$

which is easier to work with since only one Tikhonov problem is involved. Note that $\sum_{i=1}^{N}(H(i,:)x_\lambda - b_i)^2 = \|Hx_\lambda - b\|_2^2$.

Definition 2.1. The GCV function is defined by

$$\mathcal{G}(\lambda) = \frac{\|Hx_\lambda - b\|_2^2}{\mathcal{T}(\lambda)^2}, \tag{2.2}$$

where $\mathcal{T}(\lambda) = \mathrm{trace}(I - H(H^TH+\lambda^2 I)^{-1}H^T)$ is the degree of freedom and $x_\lambda = (H^TH+\lambda^2 I)^{-1}H^Tb$ is the regularization solution of (1.1).

The GCV parameter chosen method for Tikhonov regularization chooses $\lambda = \lambda_{GCV}$ as the minimizer of $\mathcal{G}(\lambda)$. An estimate of the regularization parameter can be found by evaluating the GCV function $\mathcal{G}(\lambda)$ by the singular value decomposition of H.

Lemma 2.2. *If $\{\sigma_i\}_{i=1}^N$ represents the singular values of H, then*

$$\mathcal{G}(\lambda) = \frac{\sum_{i=1}^{N}\left(\frac{\widehat{b}_i}{\sigma_i^2+\lambda^2}\right)^2}{\left(\sum_{i=1}^{N}\frac{1}{\sigma_i^2+\lambda^2}\right)^2}, \tag{2.3}$$

where $\widehat{b} = U^Tb$.

Proof. Let $H \equiv U\Sigma V^T$ be the singular value decomposition of H. By direct computations,

$$\begin{aligned} Hx_\lambda - b &= (H(H^TH+\lambda^2 I)^{-1}H^T - I)b \\ &= U(\Sigma(\Sigma^2+\lambda^2 I)^{-1}\Sigma - I)U^Tb. \end{aligned}$$

So $\|Hx_\lambda - b\|_2^2 = \sum_{i=1}^{N}\left(\frac{\lambda^2\widehat{b}_i}{\sigma_i^2+\lambda^2}\right)^2$ and $\mathcal{T}(\lambda) = \sum_{i=1}^{N}\frac{\lambda^2}{\sigma_i^2+\lambda^2}$. Thus we obtain the expression of (2.3). □

If the discrete Picard condition is satisfied and the noise is white, then the regularization parameter λ_{GCV} which minimizes the expected value of $\mathcal{G}(\lambda)$ is near the minimizer of the expected value of the predictive mean-square error $\|Hx_\lambda - b_{exact}\|_2$ ([11]). More precisely, we have the relation $\lambda_{GCV} = \lambda_{opt}(1 + o(1))$, where λ_{opt} minimizes the predictive mean-square error and $o(1) \to 0$ as $N \to \infty$. Wahba, in the same paper ([11]), gives convergence rates for the expected value of $\left\|x^{exact} - x_{\lambda_{GCV}}\right\|_2$, where $x_{\lambda_{GCV}}$ is the Tikhonov solution.

The degree of freedom $\mathcal{T}(\lambda) = N - \sum\limits_{i=1}^{N} \frac{\sigma_i^2}{\sigma_i^2+\lambda^2}$ is a slowly increasing function of λ. The following theorem sheds more light on the behavior of $\mathcal{T}$.

THEOREM 2.3. *If there is a constant ratio c between the smallest and the largest singular values, such that $\sigma_i = c\sigma_{i+1}, i = 1 \dots N$ (with $0 < c < 1$), then*

$$\nu_\lambda - \frac{1}{1-c^2} \le \mathcal{T}(\lambda) \le \nu_\lambda + \frac{1}{1-c^2}, \tag{2.4}$$

where ν_λ is the number of singular values less than λ, i.e., $\sigma_{\nu_\lambda} < \lambda \le \sigma_{\nu_\lambda+1}$.

Proof. From

$$\mathcal{T}(\lambda) = \sum_{i=1}^{N} \frac{\lambda^2}{\sigma_i^2+\lambda^2} = \sum_{i=1}^{\nu_\lambda} \left(1 - \left(1 + \left(\frac{\lambda}{\sigma_i}\right)^2\right)^{-1}\right) + \sum_{i=\nu_\lambda+1}^{N} \left(1 + \left(\frac{\sigma_i}{\lambda}\right)^2\right)^{-1},$$

the following expressions can be obtained

$$\sum_{i=1}^{\nu_\lambda} \left(1 - \left(1 + \left(\frac{\lambda}{\sigma_i}\right)^2\right)^{-1}\right) = \nu_\lambda - \sum_{i=1}^{\nu_\lambda} \left(1 + \left(\frac{\lambda}{\sigma_i}\right)^2\right)^{-1}$$

and

$$0 \le \sum_{i=1}^{\nu_\lambda} \left(1 + \left(\frac{\lambda}{\sigma_i}\right)^2\right)^{-1} \le \sum_{i=1}^{\nu_\lambda} \left(\frac{\sigma_i}{\lambda}\right)^2 = \left(\frac{\sigma_{\nu_\lambda}}{\lambda}\right)^2 \left(c^{2(\nu_\lambda-1)} + \cdots + c^2 + 1\right),$$

$$\begin{aligned} 0 \le \sum_{i=\nu_\lambda+1}^{N} \left(1 + \left(\frac{\sigma_i}{\lambda}\right)^2\right)^{-1} &\le \sum_{i=\nu_\lambda+1}^{N} \left(\frac{\lambda}{\sigma_i}\right)^2 \\ &= \left(\frac{\lambda}{\sigma_{\nu_\lambda+1}}\right)^2 \left(1 + c^2 + \cdots + c^{2(N-\nu_\lambda-1)}\right). \end{aligned}$$

Since $1 + c^2 + \cdots + c^{2(q-1)} = (1 - c^{2q})/(1 - c^2)$ where $q = \nu_\lambda$ or $N - \nu_\lambda$, the right hand sides of the above two formula are less than equal to $\frac{1}{1-c^2}$ and so (2.4) can be proved. □

The GCV method seeks to locate the transition point where $\mathcal{V}(\lambda) = \frac{\|Hx_\lambda - b\|_2^2}{\mathcal{T}(\lambda)}$ changes to a rapid increasing variance of λ. But instead of working with the function $\mathcal{V}(\lambda)$, the GCV method uses the function $\mathcal{G}(\lambda)$. The denominator $\mathcal{T}$ of $\mathcal{G}(\lambda)$ is a monotonically increasing function of λ such that $\mathcal{G}(\lambda)$ has a minimum in the transition interval $[\sigma_1, \sigma_N]$. Hence, GCV replaces the problem of locating the transition point for $\mathcal{V}$ by a numerically well defined problem to find the minimum for the GCV function $\mathcal{G}(\lambda)$. Unfortunately, in the case of the function around the unique minimum being very flat, some of numerical difficulties can arise in computing the minimum of $\mathcal{G}(\lambda)$.

3. Extension to the global GCV

In order to drive an extension of the GCV function presented in the previous section to F-norm based form for linear least squares problems (1.2) with multiple right hand sides, we define a generalization of the cross validation for the problem (1.1):

Definition 3.1. The global GCV function is defined by

$$\mathcal{G}_{\text{global}}(\lambda) = \frac{\|HX_\lambda - B\|_F^2}{[\text{trace}(I - H(H^TH + \lambda^2 I)^{-1}H^T)]^2}, \tag{3.1}$$

where $X_\lambda = (H^TH + \lambda^2 I)^{-1}H^TB$.

Since $\mathcal{G}_{\text{global}}(\lambda)$ is a nonlinear function, the minimizer usually can not be determined analytically. Some algebraic simplifications are helpful in order to evaluate this function efficiently. In particular, when the reflective boundary conditions are used, H can be diagonalized by the orthogonal two-dimensional discrete cosine transform matrix $\mathcal{C}$ and thus the following Lemma 3.2 can be obtained.

Lemma 3.2. *If $\{\rho_i\}_{i=1}^N$ represents the spectrum of H, we can rewrite $\mathcal{G}_{global}(\lambda)$ as*

$$\mathcal{G}_{\text{global}}(\lambda) = \frac{\sum_{j=1}^{s}\sum_{i=1}^{N}\left(\frac{1}{\rho_i^2+\lambda^2}[\mathcal{C}B_j]_i\right)^2}{\left(\sum_{i=1}^{N}\frac{1}{\rho_i^2+\lambda^2}\right)^2}, \tag{3.2}$$

where B_j is the j-th column of B.

Proof. To simplify the equality of (3.1), substitute a unitary spectral decomposition of H, $H = \mathcal{C}^T \Lambda_H \mathcal{C}$ when $\Lambda_H = diag(\rho_1, \rho_2, \ldots, \rho_N)$, into $HX_\lambda - B$ to get

$$HX_\lambda - B = \mathcal{C}(\Lambda_H(\Lambda_H^2 + \lambda^2 I)^{-1}\Lambda_H - I)\mathcal{C}^T B.$$

Showing that $\|Hx_\lambda - B\|_F^2 = \sum_{j=1}^{s}\sum_{i=1}^{N}\left(\frac{\lambda^2[\mathcal{C}B_j]_i}{\rho_i^2+\lambda^2}\right)^2$ and the denominator of (3.1) becomes $\left(\sum_{i=1}^{N}\frac{\lambda^2}{\rho_i^2+\lambda^2}\right)^2$, we obtain (3.2). $\square$

The regularization parameter λ satisfies $\rho_{min} \leq \lambda \leq \rho_{max}$, where ρ_{min} is the smallest eigenvalue of H and ρ_{max} is the largest eigenvalue of H. Now to find a regularization parameter λ by minimizing the global GCV function (3.2), we solve the following constrained optimization problem with the bounded constraint of λ:

$$\begin{aligned} &\min_{\lambda} && \mathcal{G}_{\text{global}}(\lambda) \\ &\text{subject to} && \rho_{min} \leq \lambda \leq \rho_{max}. \end{aligned} \tag{3.3}$$

There always exists a local minimizer of $\mathcal{G}_{\text{global}}(\lambda)$ which is a continuous function on a closed and bounded interval. To find the constrained minimum of single variable bounded nonlinear function (3.3), we can use the algorithm based on golden section search and parabolic interpolation. Golden section search is guaranteed to work in the worst possible case, and the price of this safety is slowness of convergence which is only linear([2]). At the general stage k, one has $\lambda_{k-2}, \lambda_{k-1}$, and λ_k. Let λ_{k+1} be the abscissa of the maximum ordinate of a parabola through $(\lambda_i, G(\lambda_i))$, $i = k-2, k-1, k$. This successive parabolic interpolation and optimization are repeated for the iteration $k+1$ with λ_{k-1}, λ_k, and λ_{k+1} until the error of the executive iterates gets some tolerance.

With $\lambda = \lambda_{gGCV}$, the best way to solve (1.2) numerically is to treat it as a minimization problem

$$\min_X \left\| \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix} X - \begin{pmatrix} B \\ O \end{pmatrix} \right\|_F \tag{3.4}$$

in certain situations and use the normal equations

$$(H^T H + \lambda_{gGCV}^2 I)X = H^T B. \tag{3.5}$$

The global conjugate gradient linear least squares(Gl-CGLS) method as an iterative regularization method was designed for solving large sparse systems (3.5) of equations with multiple right hand sides. The symmetric positive definite coefficient matrix $H^T H + \lambda^2 I$ can be reduced to tridiagonal matrix by the global Lanczos algorithm which constructs an F-orthogonal basis of the matrix Krylov subspace. An approximated solution can be obtained by LU factorization of the tridiagonal matrix. Let X_0 denote the initial, and define $R_0 = \begin{pmatrix} B \\ O \end{pmatrix} - \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix} X_0$, $P_0 = S_0 = \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_0$, and $\gamma_0 = (S_0, S_0)_F$. Then the Gl-CGLS iterations take the following form for $k = 0, 1, 2, ...$

1. $Q_k = \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix} P_k$, $\alpha_k = \gamma_k/(Q_k, Q_k)_F$,
2. $X_{k+1} = X_k + \alpha_k P_k$, $R_{k+1} = R_k - \alpha_k Q_k$,
3. $S_{k+1} = \begin{pmatrix} H \\ \lambda I \end{pmatrix}^T R_{k+1}$, $\gamma_{k+1} = (S_{k+1}, S_{k+1})_F$,
4. $\beta_k = \gamma_{k+1}/\gamma_k$, $P_{k+1} = S_{k+1} + \beta_k P_k$.

An essential property of Gl-CGLS iterates X_k with residual matrices R_k is that the corresponding residual matrices $S_k = \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_k$ for normal equations are orthogonal. An important consequence is that if the starting matrix X_0 is zero then the solution X_k increases monotonically with k. It can be shown from the following theorem.

THEOREM 3.3. *Let $\psi(X_k)$ denote the error function of Gl-CGLS with the global GCV :*

$$\psi(X_k) = \left(X_{LS} - X_k, (H^T H + \lambda_{gGCV}^2 I)(X_{LS} - X_k)\right)_F, \tag{3.6}$$

where X_{LS} is a solution of (3.4). Then the Gl-CGLS iterate X_k can be written as

$$X_k = X_0 + \sum_{i=0}^{k-1} \frac{\psi(X_i) - \psi(X_k)}{\left(\begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_i, \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_i\right)_F^2} \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_i,$$

and if $X_0 = O$ then

$$\|X_k\|_F^2 = \sum_{i=0}^{k-1} \left(\frac{\psi(X_i) - \psi(X_k)}{\left(\begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_i, \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_i\right)_F} \right)^2. \tag{3.7}$$

Proof. From (3.6) and $X_{k+1} = X_k + \alpha_k P_k$, $\psi(X_{k+1})$ can be expressed by

$$\psi(X_{k+1}) = \psi(X_k) - 2\alpha_k (P_k, S_k)_F + \alpha_k^2 \left(P_k, \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T S_k\right)_F. \tag{3.8}$$

The right hand side of (3.8), considered as a function in terms of α_k, has a minimum value at $\alpha_k = \left(P_k, \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T S_k \right)_F^2 / (S_k, S_k)_F$. Substituting the α_k to (3.8) we get

$$\psi(X_k) - \psi(X_{k+1}) = \left(P_k, \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T S_k \right)_F^2 / (S_k, S_k)_F = \alpha_k (S_k, S_k)_F.$$

P_k can be written by $P_k = (S_k, S_k)_F \sum_{j=0}^{k} \frac{S_j}{(S_j, S_j)_F}$, and so

$$X_k = X_{k-1} + \alpha_{k-1} P_{k-1} = X_0 + \sum_{j=0}^{k} \frac{\psi(X_i) - \psi(X_k)}{(S_j, S_j)_F}.$$

In particular, if $X_0 = O$ we get (3.7) since $\{P_k\}_{k=1,2,...}$ are mutually conjugate. □

Since the error function $\psi(X_k)$ decreases monotonically with k, we conclude that if the starting matrix of Gl-CGLS is zero, then each of the terms in (3.7) increases with k and thus the solution norm $\|X_k\|_F$ increases montonically with k. The residual norm $\|R_k\|_F$, on the other hand, decreases monotonically with k if $X_0 = O$.

The preconditioned version of (3.5) becomes

$$\Omega^{-T} \left((H^T H + \lambda_{gGCV}^2 I) \Omega^{-1} Y - H^T B \right) = O, \tag{3.9}$$

where Ω^{-T} is a preconditioning matrix and $Y = \Omega X$. The matrix $\begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix} \Omega^{-1}$ in (3.9) becomes well conditioned.

Summarizing the above process so far, the algorithm for preconditioned Gl-CGLS with the extended global GCV can be presented as follows.

Algorithm 1. Preconditioned Gl-CGLS with the global GCV

1. Determine the minimizer λ_{gGCV} for the constrained minimization problem:

$$\min_{\lambda} \mathcal{G}_{\text{global}}(\lambda) \text{ subject to } \rho_{min} \le \lambda \le \rho_{max}.$$

2. Solve $\Omega^{-T}(H^T H + \lambda_{gGCV}^2 I)X = \Omega^{-T} H^T B$ using preconditioned Gl-CGLS:
   i. $R_0 = \begin{pmatrix} B \\ O \end{pmatrix} - \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix} X_0,$
   ii. $P_0 = S_0 = \Omega^{-T} \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_0$, $\gamma_0 = (S_0, S_0)_F$,
   iii. For $k = 0, 1, 2, ...$ until convergence do
      (i) $T_k = \Omega^{-1} P_k \; Q_k = \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix} T_k$, $\alpha_k = \gamma_k / (Q_k, Q_k)_F$,

(ii) $X_{k+1} = X_k + \alpha_k T_k$, $R_{k+1} = R_k - \alpha_k Q_k$,

(iii) $S_{k+1} = \Omega^{-T} \begin{pmatrix} H \\ \lambda_{gGCV} I \end{pmatrix}^T R_{k+1}$, $\gamma_{k+1} = (S_{k+1}, S_{k+1})_F$,

(iv) $\beta_k = \gamma_{k+1}/\gamma_k$, $P_{k+1} = S_{k+1} + \beta_k P_k$.

For the reflective boundary condition H has the structures of block Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks which can be diagonalized by two dimensional discrete cosine transformation matrix $\mathcal{C}$. So the preconditioner Ω in (3.9) is set to $\Omega = \mathcal{C}^T(|\Lambda_H|^2 + \lambda^2 I)^{1/2}\mathcal{C}$.

4. Numerical experiments

Employing the global GCV in preconditioned Gl-CGLS method for solving image restoration problems with four common test images, we investigated numerical results to illustrate the effectiveness of the regularization parameters chosen from the minimization of the global GCV function. For comparison purposes, various possible regularization parameters chosen experimentally by attempting to minimize the relative accuracy are used in each test. In order to get the local minimizer λ_{gGCV} of $\mathcal{G}_{\text{global}}(\lambda)$, (3.3) was solved by the matlab function *fminbnd*, a method based on the golden section search and the parabolic interpolation. Preconditioned Gl-CGLS iteration was stopped once the current residual satisfies the criteria $\|R_k\|_F \,/\, \|R_0\|_F \le tol$, where tol is set to 10^{-4}.

In Tables 1 and 2, the numerical results of the preconditioned Gl-CGLS method applied to four image restorations are presented. The number of iterations, the execution time for preconditioned Gl-CGLS, PSNR, and the relative accuracy are shown. The peak-to-signal ratio(PSNR) is defined as $10\log_{10}\left(\frac{255^2}{\frac{1}{mn}\sum_{i,j}(x^*_{i,j}-\widehat{x}_{i,j})^2}\right)$, where $x^*_{i,j}$ and $\widehat{x}_{i,j}$ denote the pixel value of the original image and restored image respectively. PSNR is commonly used to measure the quality of reconstruction of lossy compression codecs. Typical values for PSNR in lossy image are between 30 and 50 dB, where higher is better. The relative accuracy

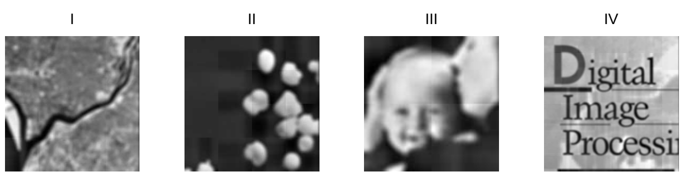

FIGURE 1. *Out-of-focus* blur and noisy images

FIGURE 2. Reconstructed images

$\|x^* - \widehat{x}\|/\|x^*\|$ shows how well the true image has been approximated.

Four test images named by I through IV are degraded by *out-of-focus* blur and Gaussian blur respectively with adding Gaussian noises. The *out-of-focus* blur arises when the lens is out of focus and its point spread function $H(x, y, x, y')$ is $(\pi r^2)^{-1}$ if $\sqrt{(x-x')^2 + (y-y')^2} \le r$ and 0 else, where the parameter r characterizes the defocus. The radius of defocus r is set to 4 and 5 in our test. Figure 1 shows four test images, blurred by *out-of-focus* blur and Gaussian noise being added. The sizes of degraded images are 256-by-256(I, II), 128-by-128(III), and 512-by-512(IV). Using reflective boundary condition, these images are divided into the collection of 16, 4, and 64 small block images with the sizes of 64-by-64 respectively. The reconstructed images by preconditioned Gl-CGLS with the global GCV are given in Figure 2.

Preconditoned Gl-CGLS with	Test image	Iteration /cpu(sec)	Relative accuracy	PSNR
global GCV	I	444/ 82.19	0.036152	28.2821
	II	236/ 43.93	0.020535	37.5972
	III	264/ 19.58	0.020399	37.9828
	IV	206/ 156.29	0.012783	29.8394
$\lambda = 0.1$	I	899/ 166.42	0.062700	23.4994
	II	892/ 165.16	0.083031	25.4623
	III	1074/ 80.78	0.042076	31.6942
	IV	612/ 454.96	0.039615	20.0149
$\lambda = 0.8$	I	442/ 82.07	0.036121	28.2895
	II	241/ 44.42	0.020551	37.5906
	III	261/ 19.48	0.020438	37.9660
	IV	209/ 155.79	0.012646	29.9326

TABLE 1. *Out-of-focus* blur

Table 1 presents the performance results of preconditioned Gl-CGLS method with regularization parameters chosen from both of the extended global GCV and the numerical experiments having the relative accuracy as small as possible. The table shows that preconditioned Gl-CGLS

Preconditoned Gl-CGLS with	Test image	Iteration /cpu(sec)	Relative accuracy	PSNR
global GCV	I	552/ 105.79	0.037913	27.8689
	II	340/ 63.58	0.020174	37.7515
	III	1210/ 60.59	0.020234	38.0533
	IV	196/ 155.65	0.011999	30.3893
$\lambda = 0.008$	I	1617/ 302.82	0.025249	31.3998
	II	2022/ 384.63	0.020944	37.4259
	III	2372/ 107.65	0.021427	37.5555
	IV	1028/ 759.56	0.019557	26.1461
$\lambda = 0.01$	I	1366/ 252.49	0.026071	31.1215
	II	1682/ 309.39	0.020421	37.6458
	III	1975/ 89.00	0.019102	38.5531
	IV	885/ 655.53	0.016643	27.5474
$\lambda = 0.8$	I	543/102.96	0.037891	27.8740
	II	336/ 63.18	0.020162	37.7564
	III	381/ 17.64	0.022922	36.9700
	IV	193/145.93	0.011990	30.3956

Table 2. Gaussian blur

with the global GCV is 2.73 times more efficient on average than preconditioned Gl-CGLS with $\lambda = 0.1$ in regard to the relative accuracy for all four images. PSNR of preconditioned Gl-CGLS with the global GCV also increases for all cases to 134% on average of PSNR of preconditioned Gl-CGLS with $\lambda = 0.1$ which generally indicates that the reconstruction is of higher quality. For $\lambda = 0.8$ which draws the less relative accuracy among λs tested by many numerical trial and error, preconditioned Gl-CGLS has reconstructed the closer images to the true images than preconditioned Gl-CGLS with the global GCV for only two images(II, III).

Table 2 shows the numerical results for the same number of blocks of the original images degraded by Gaussian blur and Gaussian white noises added with standard deviation of 0.001. The performances of preconditioned Gl-CGLS with the global GCV for three images(II, III, IV) give better reconstructions than those of preconditioned Gl-CGLS with $\lambda = 0.008$, especially the relative accuracy for the image IV decreases about 1.63 times and PSNR increases about 1.16 times. As shown in Table 2, however, $\lambda = 0.8$ for preconditioned Gl-CGLS method is the best choice whose performance is a little better than the results with parameter chosen from the global GCV. Since $\lambda = 0.8$ was obtained from trial and error, the regularization parameter from the global GCV can be considered as cost effective in selecting and reliable.

The tables 1 and 2 show also that preconditioned Gl-CGLS with the global GCV requires less of iterations and cpu times for $\lambda = 0.1, 0.008$ and 0.01.

Consequently, we conclude that the preconditioned Gl-CGLS method with the extended global GCV for large image restoration problems is stable since the choice of regularization parameter from the global GCV has better performances than any other known determinations from many numerical experiments shown in this paper. The exploration of weighted techniques in the global GCV will be considered as our next task.

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